arXiv:gr-qc/9912025v1 8 Dec 1999

Dimensionally Democratic Calculus and Principles of Polydimensional Physics *

William M. Pezzaglia Jr. †
Department of Physics
Santa Clara University, CA

February 7, 2008

Abstract

A solution to the 50 year old problem of a spinning particle in curved space has been recently derived using an extension of Clifford calculus in which each geometric element has its own coordinate. This leads us to propose that all the laws of physics should obey new polydimensional metaprinciples, for which Clifford algebra is the natural language of expression, just as tensors were for general relativity. Specifically, phenomena and physical laws should be invariant under local automorphism transformations which reshuffle the physical geometry. This leads to a new generalized unified basis for classical mechanics, which includes string theory, membrane theory and the hypergravity formulation of Crawford[J. Math. Phys., **35**, 2701-2718 (1994)]. Most important is that the broad themes presented can be exploited by nearly everyone in the field as a framework to generalize both the Clifford calculus and multivector physics.

Summary of presentation, submitted to the *Proceedings of the 5th International Conference on Clifford Algebras and their Applications in Mathematical Physics, Ixtapa-Zihuatanejo, MEXICO, June 27-July 4, 1999*, R. Ablamowicz and B. Fauser eds.

*Financial Support for attending the conference provided by Santa Clara Univ.

†wpezzag@clifford.org

I. Introduction

Taking an existing equation and generalizing it over quaternions or Clifford numbers is certainly a way of doing new mathematics. It is important however to understand that this seldom leads to new physics (for example complexifying Newton's law of gravitation is meaningless). Reformulating existing physical laws with a new mathematical language will not lead to new principles nor new physics. Only by generalizing principles can we hope to do something new. However, because Clifford algebra[11] encodes the structure of the underlying geometric space, we see possible bigger patterns emerge. Specifically in the description of a spinning particle the equations of motion are invariant under a non-dimensional preserving *polydimensional* transformation which rotates between vector momentum and bivector spin. We therefore propose that 'what is a vector' is *Dimensionally Relative* to the observer's frame, and that the universe is fully *Polydimensionally Isotropic* in that there is no absolute 'direction' which we can assign 'vector' geometry over bivector, trivector, etc.

This forces us to propose a fully *Dimensionally Democratic* Clifford calculus, in which each geometric element has its own coordinate in a *Clifford manifold*[3]. In particular we show the utility of this concept in treating the classical spinning particle in several scenarios. A new action principle is proposed in which particles take paths which minimize the sum of the linear distance traveled combined with the bivector area swept out by the spin. In curved space, the velocity of the variation is not the variation of the velocity, leading to a new derivation of the Papapetrou equations[14] as autoparallels in the Clifford manifold. This leads us to propose that the physical laws might be *Metomorphic Covariant* under general automorphism transformations which reshuffle the geometry.

II. Relative Dimensionalism

Most physicists tend to be absolute in their association of physical quantities to geometric entities. For example, mass is a scalar while force a vector. The introduction of Einstein's relativity however caused a shift in dimensional interpretation in that 'the world' is not three-dimensional but a four-dimensional spacetime continuum. In the 'old-fashion' three dimensional viewpoint, energy was a scalar, but in the new 4D paradigm it is the fourth component of the momentum vector. Certainly many physicists will share the opinion that the new 4D viewpoint is right, and the old 3D view is incorrect. Yet let us consider for example the recent book by Baylis[1] in which he has a complete treatment of electrodynamics and special relativity using *paravectors* (defined as the Clifford aggregate of a scalar plus three-vector). Is he wrong to call time the scalar part of a paravector instead of calling it the fourth component of a vector?

Consider also that the even subalgebra of the 16 element Clifford algebra associated with 4D spacetime can be interpreted as the Clifford algebra of a 3D space. Three of the planes of 4D are reinterpreted as basis vectors in the 3D space, while the four-volume is reinterpreted as a three-volume. So if you grab

a particular geometric element, is it a vector or a plane? We suggest that there is no absolute right or wrong answer. We postulate a new principle of **relative dimensionalism**: *that the geometric rank an observer assigns to an object is a function of the observer's frame of reference* (or perhaps state of conciousness). There is no "absolute" dimension that one can assign to a geometric object. Further we consider transformations which reshuffle the basis geometry (e.g. vector line replaced by bivector plane), yet leave sets of physical laws invariant. One application provides a new treatment of the classical spinning particle, showing that the mechanical mass is enhanced by the spin motion.

A. Review of Special Relativity

It is useful to see how paradigm shifts in the concept of the dimensional nature of space have impacted the formulation of physical laws in the past for clues as how to proceed with newer ideas. The *prima facie* example is how things changed with the introduction of special relativity. Quantities such as time and energy, that were previously defined as scalars (in a 3D formulation) now are identified as fourth components of vectors in four-dimensional Minkowski spacetime. Let us consider what is gained by using the higher dimensional concept.

1. Unification of Phenomena

The most obvious advantage of using vectors is that one can replace a set of physical equations by a single vector equation. Let us consider the application of four-vectors in electrodynamics. The 3D scalar work-energy law and 3D vector force law,

$$\dot{\mathcal{E}} = e\,\vec{\mathbf{E}}\cdot\vec{\mathbf{v}}\ , \qquad \dot{\vec{\mathbf{P}}} = e\,\left(\vec{\mathbf{E}} + \vec{\mathbf{v}}\times\vec{\mathbf{B}}\right)\ , \tag{1ab}$$

can be combined into one single equation,

$$\dot{p}^{\mu} = \left(\frac{e}{mc}\right) p_{\nu}\, F^{\mu\nu}\ , \tag{2}$$

using 4D vectors and tensors (c is the speed of light and the dot represents differentiation with respect to time). Certainly the adoption of the four-dimensional viewpoint has notational economy, and provides insight that the work-energy theorem (1a) is simpy the fourth aspect of the vector force law (2). However, philosophically one can ask if the 4D viewpoint is any more correct than the 3D equations as they describe the same phenomena. Since special relativity was originally formulated without the concept of Minkowski spacetime, it is convenient, but apparently not necessary to adopt the paradigm shift from 3D to 4D. Hence we are being purposely dialectic in raising the question whether one can make an absolute statement about the dimensional nature of a physical quantity such as time. Can we state (measure) that time is a part of a four-vector (as opposed to a 3D scalar), or is this relative to whether one adopts a 3D or 4D world view, hence relative to the observer's dimensional frame of reference?

2. Lorentz Transformations

In classical physics the fundamental laws must be invariant under rotational displacements because it is postulated that the universe is *isotropic* (has no preferred direction). When one formulates laws with vectors (which are inherently coordinate system independent), isotropy is 'built in' without needing to separately impose the condition. Hence (Gibbs) vectors are a natural language to express classical (3D) physical laws because they naturally encode isotropy.

Einstein further postulated the metaprinciple that motion was relative; that there is no absolute preferred rest frame to the universe. This coupled with the postulate that the speed of light is the same for all observers leads to the principle that the laws of physics must be invariant under Lorentz transformations (which connect inertial frames of reference). A more geometric interpretation is to see that Lorentz transformations are just rotations in 4D spacetime, hence the principle of relative motion is an extension of the metaprinciple of isotoropy to four-space.

Lorentz transformations, expressed in four-space, preserve the rank of geometry (rotates a four-vector into another four-vector). In contrast, Baylis[1] would write the Lorentz boost (in the z direction with velocity v) of the *momentum paravector* in 3D Clifford algebra as,

$$\mathbf{1}\mathcal{E}'/c + \vec{\mathbf{P}}' = \mathcal{R}\left(\mathbf{1}\mathcal{E}/c + \vec{\mathbf{P}}\right)\mathcal{R}^{\dagger} \ , \tag{3}$$

where c is the speed of light, and the transformation operator: $\mathcal{R} = \exp(-\hat{\mathbf{e}}_3\beta/2)$, where β the rapidity related to the velocity: $\tanh(\beta) = v/c$ and $\hat{\mathbf{e}}_3$ is the unit vector in the z direction. As a consequence, in this 3D perspective, what is pure scalar (e.g. $\mathcal{E}$ =energy) to one observer is part scalar, part vector to another observer. Lorentz transformations, in this 3D viewpoint, are NOT dimensional preserving. We choose to classify such transformations as *geometamorphic* or 'polydimensional'.

3. Invariant Moduli

In 3D space the length (magnitude) of a vector (e.g. electric field or momentum) is invariant under rotations. Under Lorentz transformations (4D rotations), the modulus of the four-vector is invariant,

$$\parallel \mathbf{p} \parallel^2 \equiv p_\mu p^\mu = \mathcal{E}^2/c^2 - \parallel \vec{\mathbf{P}} \parallel^2 \ . \tag{4}$$

Reinterpreted with a 3D viewpoint, the invariant quantity of the Lorentz transformation (3) is the difference between the square of the scalar energy minus the magnitude of the 3D momentum vector. Neither the modulus of the 3D scalar, nor 3D vector is independently invariant under these transformations.

The modulus of the momentum four-vector (4a) is defined to be the *rest mass* of the particle: $m_0 \equiv c^{-1} \parallel \mathbf{p} \parallel$. When in motion, the mechanical mass of the particle (e.g. in definition of momentum: $p = mv$) increases by its kinetic energy content. This is described by the *Lorentz dilation factor* γ,

$$m \equiv \gamma\, m_0 \ , \tag{5a}$$

$$\gamma \equiv \cosh\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \sqrt{1 + \left(\frac{\| \vec{\mathbf{P}} \|}{m_0 c}\right)^2} \ . \tag{5b}$$

B. Automorphism Invariance

Transformations that preserve some physical symmetry often lead to a conservation law of physics. For example, displacements of the origin leave Newton's laws unchanged, leading to a derivation of the conservation of linear momentum. Central force problems (e.g. gravitational field around a spherical star) have rotational invariance, leading to conservation of angular momentum for orbits. When formulating physics with Clifford algebra, we should ask just what new symmetries are inherently encoded in the structure of the algebra, and what (if any) new physical laws they may imply.

1. Matrix Representation Invariance

Physicists usually first encounter Clifford algebras in quantum mechanics in the form of Pauli, Majorana and Dirac 'spin' matrices. The matrix representation for example of the four generators γ^μ of the Majorana algebra is arbitrary. Hence it is obvious that the physics must be invariant under a change of the matrix representation of the algebra. A change in representation can be reinterpreted as a global rotation of the spin space basis spinors. Requiring 'spin space isotropy' (no preferred direction in spin space) leads to the physical principle of conservation of quantum spin.

2. Algebra Automorphisms

Its possible however to avoid talking about the matrix representation entirely. The more general concept is an algebra automorphism, which is a transformation of the basis generators γ_μ of the algebra which preserves the Clifford structure,

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu} \ , \tag{6}$$

where $g_{\mu\nu}$ is the spacetime metric. For example, consider the following orthogonal transformation on any element $\mathcal{Q}$ of the Clifford algebra,

$$\mathcal{Q}' = \mathcal{R}\, \mathcal{Q}\, \mathcal{R}^{-1} \ , \tag{7}$$

$$\mathcal{R}(\phi^\mu) \equiv \exp(-\gamma_\mu\, \phi^\mu/2) \ , \qquad \mu = 1, 2, 3, 4. \tag{8}$$

Proposing local covariance of the Dirac equation under such automorphism transformations is one path to gauge gravity and grand unified theory[4].

3. Polydimensional Isotropy

If the elements γ_μ are interpreted geometrically as basis vectors, then (8) reshuffles geometry. For example, when $\phi^4 = \pi/2$, eq. (7) causes the permutation,

$$\gamma_j \Longleftrightarrow \gamma_4 \gamma_j \ , \quad j = 1, 2, 3, \tag{9a}$$

$$\gamma_1\gamma_2\gamma_3 \Longleftrightarrow \gamma_4\gamma_1\gamma_2\gamma_3 \ , \tag{9b}$$

which exchanges three of the vectors with their associated timelike bivectors. What is a 1D vector in one "reference frame" is hence a 2D plane in another. The transformation (8) thus "rotates" vectors into planes. Another example would be the transformation generated by (7) for: $\mathcal{R} = \exp(\hat{\epsilon}\,\phi/2)$, where $\hat{\epsilon} \equiv \gamma_1\gamma_2\gamma_3\gamma_4$. When $\phi = \pi/2$ this causes the duality transformation,

$$\gamma_\mu \Longleftrightarrow \hat{\epsilon}\,\gamma_j \ , \quad \mu = 1, 2, 3, 4, \tag{10}$$

exchanging vectors for dual trivectors (the rest of the algebra is unchanged).

If we feel that Clifford algebra is the natural description of a geometric space, then we must ask whether these algebra automorphisms have physical interpretation. We suggest that there is a 'higher octave' to the metaprinciple of the isotropy of space. We propose the **principle of polydimensional isotropy**: that *there is no absolute or preferred direction in the universe to which one can assign the geometry of a vector*. Just as which direction you choose to call the z-axis is arbitrary, it is also arbitrary just which geometric element you call the basis vector in the z-direction. Another observer may make an entirely different selection. This suggests perhaps that we should require the laws of physics to be invariant under such automorphic transformations (which will be discussed in more detail in Section IV).

C. Polydimensional Formulation of Physics

If we embrace the new principle that space is polydimensionally isotropic, then we should consider if transformations that reshuffle the basis geometry leave certain sets of physical laws invariant. Our development will attempt to parallel that which happened in the transition from classical 3D physics to 4D special relativity.

1. Unification with Clifford Algebra

Just as four-vectors allowed us to unify two equations into one, the language of Clifford algebra allows for further notational economy. Consider that a classically spinning point charged particle obeys the torque equation of motion[18],

$$\dot{S}^{\mu\beta} = \left(\frac{e}{mc}\right)\left(F^{\mu}{}_{\nu}\, S^{\nu\beta} - F^{\beta}{}_{\nu}\, S^{\nu\mu}\right) = \left(\frac{e}{mc}\right)(\mathbf{F} \otimes \mathbf{S})^{\mu\beta} \ . \tag{11}$$

This and eq. (2) can be written in the single statement,

$$\dot{\mathcal{M}} = \left(\frac{e}{2mc}\right)[\mathbf{F}, \mathcal{M}] \ , \tag{12a}$$

where $\mathbf{F} \equiv \frac{1}{2}F^{\mu\nu}\,\hat{\mathbf{e}}_\mu \wedge \hat{\mathbf{e}}_\nu$ is the electromagnetic field bivector and $\hat{\mathbf{e}}_\mu$ are the basis vectors of the geometric space which obey the same Clifford algebra rules

eq. (6). The *momentum polyvector* is defined as the multivector sum of the vector linear momentum and the bivector spin momentum,

$$\mathcal{M} \equiv p^\mu \hat{\mathbf{e}}_\mu + \frac{1}{2\lambda} S^{\mu\nu} \hat{\mathbf{e}}_\mu \wedge \hat{\mathbf{e}}_\nu \ , \tag{12b}$$

where λ is some fundamental length scale constant (to be interpreted in the next section) that allows us to add the quantities with different units in analogy to the use of c in eq. (3). The ability to add different ranked (dimensional) geometries is the notational advantage of Clifford geometric algebra over standard tensors. Mathematically, (12a) allows one to *simultaneously* obtain solutions to both equations (2) and (11): $\mathcal{M}(\tau) = \mathcal{R}\,\mathcal{M}(0)\,\mathcal{R}^{-1}$, where the rotation operator,

$$\mathcal{R}(\tau) = \exp\left(\frac{e}{4mc}\,\hat{\mathbf{e}}_\mu \wedge \hat{\mathbf{e}}_\nu \int^\tau d\tau' F^{\mu\nu}\left[x(\tau')\right]\right) \ , \tag{13}$$

involves a path (history) dependent integral, hence the solution is formal.

2. Polydimensional Invariance

Equation (12a) is manifestly covariant under automorphism transformations. Specifically, the set of equations (2) and (11) are invariant under the automorphism transformations generated by (8). For example, $\phi^4 = \pi/2$ in (8) causes a trading between momentum and mass moment of the spin tensor,

$$\lambda\, p_j \Longleftrightarrow S_{4j} \ . \tag{14}$$

It is not at all clear what physical interpretation to ascribe to the two frames of reference. A radical assertion of the *principle of relative dimensionalism* would be to propose that what is a vector to one observer is a bivector to another, and that they would partition the polymomentum (12b) into momentum and spin portions differently. *What is spin to one would be momentum to the other.*

Just as rotational invariance led to conservation of angular momentum, we might ask just what is the conserved quantity associated with this new symmetry transformation. This will be addressed in Section III.B.3 below.

3. The Quadratic Form of a Polyvector

We define the modulus of the polyvector eq. (12b) to be the square root of the scalar part of the square of the polyvector,

$$\parallel \mathcal{M} \parallel^2 \equiv \ p_\mu\, p^\mu - \frac{1}{2\lambda^2} S_{\mu\nu}\, S^{\mu\nu} \ . \tag{15}$$

This quadratic form is invariant under the rotation of vectors into bivectors generated by (8). In the $(---+)$ metric signature, we define the modulus of the momentum polyvector to be the *bare mass*: $m_0 \equiv c^{-1} \parallel \mathcal{M} \parallel$. This implies

that the mechanical mass (modulus of the momentum) is NOT invariant under these transformations, but has been enhanced by the spin energy content,

$$m \equiv c^{-1} \parallel \mathbf{p} \parallel = m_0 \sqrt{1 + \frac{S^{\mu\nu} S_{\mu\nu}}{2\,(m_0 c \lambda)^2}}\ , \tag{16a}$$

in analogy to (5ab). What we have described in (16a), by simple geometric construction, is a familiar result, laboriously obtained by Dixon[6] in the mechanical analysis of extended spinning bodies. Expanding (16a) non-relativistically,

$$mc^2 \simeq m_0 c^2 + \left(\frac{\vec{\mathbf{S}}^2}{2 m_0 \lambda^2} \right) + \dots\ , \tag{16b}$$

where $\vec{\mathbf{S}}^2 \equiv (S_{12})^2 + (S_{23})^2 + (S_{31})^2$, one sees that λ is consistent with the *radius of gyration* of a classical extended particle such that its moment of inertia is $\simeq m_0 \lambda^2$. Hence the correction to the mass is due to the rotational kinetic energy.

III. Dimensional Democracy

In quaternionic analysis, a coordinate is given to each of the four elements. Reinterpreted as a Clifford algebra, it would be as if one has given a coordinate to each of the two vector directions, one to the plane, and one to the scalar. We now propose that each geometric element of the Clifford algebra democratically has its own conjugate coordinate. Further *the physical laws should be multivectorial, with each geometric component meaningful*. For example, our polymomenta (12) gives the (vector) linear momenta and (bivector) spin momenta equal importance, both contributing the the modulus of the vector. This suggests a generalized action principle that particles take the paths which minimize the sum of the linear distance traveled combined with the bivector area swept out. This simple geometric idea gives a new derivation of the spin enhanced mass described by the Dixon equation (15), as well as proposals for new quantum equations.

A. Review of Classical Relativistic Mechanics

In ancient times, Heron of Alexandria showed that light reflecting off a mirror would take the path of least distance between two endpoints. The generalized concept is that classical particles will follow paths of least spacetime distance between endpoints, even when the space is curved by gravity.

1. Time Contributes to Distance

The measure of distance between two points in flat spacetime is,

$$c^2 d\tau^2 \equiv c^2 dt^2 - \left(dx^2 + dy^2 + dz^2 \right) = dx^\alpha dx^\beta\, g_{\alpha\beta}\ , \tag{17a}$$

where affine parameter τ is commonly called the *proper time*. If we adopt the 3D viewpoint, we are combining (in quadrature) the 'scalar' time displacement with the 'vector' path displacement, utilizing a fundamental constant c (the speed of light) to combine the quantities which have different units. The metric tensor $g_{\alpha\beta}$ in flat space is diagonal with elements $(-1,-1,-1,+1)$ such that the fourth time component has the opposite signature of the spatial parts. To generalize for curved space, $g_{\alpha\beta}(x^\sigma)$ becomes a function of spacetime position.

Dividing (17a) by dt^2 recovers the *Lorentz dilation factor* eq. (5b) in terms of the nonrelativistic velocity,

$$\gamma \equiv \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} . \tag{17b}$$

2. Euler-Lagrange Equations of Motion

In simplest form, to obtain the equations of motion, one chooses the special path between fixed endpoints for which the *action integral* [which is based upon the quadratic form eq. (17a)],

$$\mathcal{A} \equiv \int m_0 c \; d\tau = \int \mathcal{L} d\tau = \int m_0 c \sqrt{u^\alpha \, u^\beta \, g_{\alpha\beta}(x^\sigma)} \; d\tau \ , \tag{18}$$

is an *extremum*. The integrand $\mathcal{L}(\tau, x^\alpha, \dot{x}^\alpha)$ is called the *Lagrangian*, which is generally a function of the coordinates and their velocities relative to the proper time: $u^\alpha \equiv \dot{x}^\alpha = dx^\alpha/d\tau$, where $x^4 \equiv ct$, hence $u^4 = \dot{x}^4 = c\gamma$.

Each coordinate x^α has a canonically conjugate momentum p_α defined,

$$p_\mu \equiv \frac{\delta \mathcal{L}}{\delta u^\mu} = m_0 u_\mu = m_0 \dot{x}_\mu \ . \tag{19a}$$

For our relativistic Lagrangian (18) these obey eq. (4). When reparameterized in terms of the more familiar observer's time $t = x^4/c$,

$$\texttt{Momentum:} \quad P_j \equiv \frac{\delta \mathcal{L}}{\delta \dot{x}^j} = m_0 \dot{x}_j = m \, v_j \ , \quad j = 1, 2, 3, \tag{19b}$$

$$\texttt{Energy:} \quad \mathcal{E} \equiv c \frac{\delta \mathcal{L}}{\delta \dot{x}^4} = c \, m_0 \, \dot{x}_4 = m \, c^2 \ , \tag{19c}$$

it is easy to show that the 3D part of the momentum $P_j = m v_j$ has mass m which is enhanced by the energy content according to (5ab).

Applying Hamilton's *Principle of Least Action*, one considers the total variation of the Lagrangian with respect to a variation in path (and velocity),

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta x^\alpha} \, \delta x^\alpha + \frac{\delta \mathcal{L}}{\delta \dot{x}^\alpha} \, \delta \dot{x}^\alpha \ . \tag{20a}$$

To get the equations of motion as that part proportional to a variation in the path only, the last term involving the variation of the velocity is integrated by parts,

$$\frac{\delta \mathcal{L}}{\delta \dot{x}^\alpha} \delta \dot{x}^\alpha \ \equiv \ p_\alpha \delta \dot{x}^\alpha \ = \ \frac{d}{d\tau} \left(p_\alpha \delta x^\alpha\right) - \dot{p}_\alpha \delta x^\alpha + p_\alpha \left(\delta \dot{x}^\alpha - \frac{d}{d\tau} \delta x^\alpha\right) \ . \tag{20b}$$

The total derivative term does not contribute if the variation in path has fixed endpoints. Substituting into (20a) and setting equal to zero, one obtains the generalized Euler-Lagrange equations of motion[8],

$$\left(\dot{p}_\alpha - \frac{\delta\mathcal{L}}{\delta x^\alpha}\right)\delta x^\alpha = p_\beta\left(\delta\dot{x}^\beta - \frac{d}{d\tau}\delta x^\beta\right) \ . \tag{20c}$$

In most elementary mechanics texts the terms on the right are argued to vanish because it is assumed the variation of the velocity is the same as the velocity of the variation. However when coordinates are path dependent (non-holonomic), $\delta\mathbf{d} \neq \mathbf{d}\delta$ because derivatives will not commute[8, 13]. For example, in a rotating coordinate system,

$$\left(\delta\dot{x}^\beta - \frac{d}{d\tau}\delta x^\beta\right) = \delta x^\alpha\,\omega_\alpha^{\ \beta} \ . \tag{20d}$$

This would introduce an $\vec{\omega}\times\vec{\mathbf{P}}$ pseudoforce term on the right side of the equation of motion eq. (20c).

3. Rotating Coordinate Systems

The principle of isotropy states that there is no preferred direction in spacetime. Hence the laws of physics must be invariant under local Lorentz transformations (which include spatial rotations). Hence we can invent a new "body frame" coordinate system which has time dependent basis vectors $\mathbf{e}_\mu(\tau)$ that are related to the fixed "lab frame" basis $\hat{\mathbf{e}}_\mu$ by a time dependent orthogonal transformation,

$$\mathbf{e}_\mu(\tau) = \mathcal{R}\,\hat{\mathbf{e}}_\mu\,\mathcal{R}^{-1} \ , \tag{21a}$$

$$\mathcal{R}(\tau) = \exp\left(-\frac{1}{4}\hat{\mathbf{e}}_{\mu\nu}\,\Theta^{\mu\nu}(\tau)\right) \ . \tag{21b}$$

The cartesian angular displacement coordinates $\Theta^{\mu\nu}$ uniquely describe the orientation state of the body frame at the particular time. However, the angular velocity bivector $\underline{\omega}$ of the frame is NOT given by the time derivatives of these coordinates, rather its defined[10],

$$\underline{\omega} \equiv (1/2)\omega^{\mu\nu}\,\hat{\mathbf{e}}_{\mu\nu} \equiv -2\,\mathcal{R}^{-1}\dot{\mathcal{R}} \quad \neq \quad (1/2)\,\dot{\Theta}^{\mu\nu}\,\hat{\mathbf{e}}_{\mu\nu} \ . \tag{22}$$

The difficulty is that rotations (Lorentz transformations) do not commute, hence the final state of the body frame is a function of path history. This was also the case in the electrodynamic problem presented earlier in eq. (13).

One can invent some new *quasi-coordinates*: $\theta^{\mu\nu}$, for which the angular velocity IS given by their time derivative, $\omega^{\mu\nu} \equiv \dot{\theta}^{\mu\nu}$, (see Greenwood[9]). Unfortunately, these new coordinates are non-integrable (path dependent) and hence *non-holonomic* such that $(\delta\mathbf{d} - \mathbf{d}\delta)\theta^{\mu\nu} \neq 0$. The advantage of resorting to this complexity is that the Lagrangian and (generalized) Euler-Lagrange equations have the same form in both the body frame and lab frame[8]. Further, we can show by the chain rule,

$$\mathcal{R}\underline{\omega} = -2\dot{\mathcal{R}} = -2\left(\frac{1}{2}\dot{\theta}^{\mu\nu}\frac{\partial\mathcal{R}}{\partial\theta^{\mu\nu}}\right) \ , \tag{23a}$$

that the tangent bivectors are given by the derivatives of the rotation operator,

$$2\frac{\partial \mathcal{R}}{\partial \theta^{\mu\nu}} = -\mathbf{e}_{\mu\nu}\mathcal{R} = -\mathcal{R}\hat{\mathbf{e}}_{\mu\nu} \ . \tag{23b}$$

The differential and variation of the rotation operator are hence,

$$\mathbf{d}\mathcal{R} = -\frac{1}{2}\mathcal{R}\, d\theta^{\mu\nu}\, \hat{\mathbf{e}}_{\mu\nu} \ , \qquad \delta\mathcal{R} = -\frac{1}{2}\mathcal{R}\, \delta\theta^{\mu\nu}\, \hat{\mathbf{e}}_{\mu\nu} \ . \tag{24ab}$$

We assume that since $\mathcal{R}$ defines the state of the body *independent of the particular coordinate parametrization*, that $\delta(\mathbf{d}\mathcal{R}) = \mathbf{d}(\delta\mathcal{R})$. Explicitly taking the variation δ of eq. (24a), and setting it equal to the differential $\mathbf{d}$ of eq. (24b) we obtain,

$$(\delta\mathbf{d} - \mathbf{d}\delta)\, \frac{1}{2}\theta^{\mu\nu}\hat{\mathbf{e}}_{\mu\nu} = \mathcal{R}^{-1}\left(\mathbf{d}\mathcal{R}\, \delta\underline{\theta} - \delta\mathcal{R}\, \mathbf{d}\underline{\theta}\right) = \frac{1}{2}\left[\mathbf{d}\underline{\theta}, \delta\underline{\theta}\right] \ . \tag{25a}$$

In component form, we see that for rotations, the variation of the angular velocity bivector is <u>not</u> the velocity of the angular variation bivector,

$$\left(\delta\omega^{\mu\nu} - \frac{d\delta\theta^{\mu\nu}}{d\tau}\right) = (\underline{\omega} \otimes \underline{\delta\theta})^{\mu\nu} = \delta^{\mu\nu}_{\alpha\beta}\, \omega^{\alpha}_{\ \sigma}\, \delta\theta^{\sigma\beta} \ . \tag{25b}$$

B. Polydimensional Mechanics

If we fully embrace the concept of *relative dimensionalism*, then we must recognize that what one observer labels as a 'point' in spacetime with vector coordinates (x, y, z, t) may be seen as an entirely different geometric object by another. This suggests that perhaps we should formulate physics in a way which is completely *dimensionally democratic* in that all ranks of geometry are equally represented.

1. The Clifford Manifold

We propose therefore that 'the world' is not the usual four-dimensional manifold, but instead a fully *polydimensional continuum*, made of points, lines, planes, etc. Each event Σ is a geometric point in a *Clifford manifold*[3], which has a coordinate q^A associated with each basis element $\mathbf{E}_A$ (vector, bivector, trivector, etc.). Our definition of the "Clifford Manifold" is hence broader than the original proposal by Chisholm and Farwell[3] in that we have been "dimensionally democratic" in giving a coordinate to each geometric degree of freedom. The *pandimensional differential* in the manifold would be,

$$d\Sigma \equiv \mathbf{E}_A dq^A = \mathbf{e}_\mu dx^\mu + \frac{1}{2\lambda}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta\, da^{\alpha\beta} + \frac{1}{6\lambda^2}\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\sigma\, dV^{\alpha\beta\sigma} + \ldots \ , \tag{26a}$$

where in Clifford algebra it is perfectly valid to add vectors to planes and volumes (parameterized by the antisymmetric tensor coordinates $dx^\mu, da^{\alpha\beta}, dV^{\alpha\beta\sigma}$ respectively).

In analogy to (15), we propose that the quadratic form of the Clifford manifold would be the scalar part of the square of (26a),

$$d\kappa^2 = \| d\Sigma \|^2 \equiv dx^\mu dx_\mu + \frac{1}{2\lambda^2} da^{\alpha\beta} da_{\beta\alpha} + \frac{1}{6\lambda^4} dV^{\alpha\beta\sigma} dV_{\sigma\beta\alpha} + \ldots \quad . \tag{26b}$$

The fundamental length constant λ must be introduced in eq. (26ab) in order to add the bivector 'area' coordinate contribution to the vector 'linear' one. In analogy to (17a) this new quadratic form suggests we define a new affine parameter $d\kappa = \| d\Sigma \|$ which we will use to parameterize our polydimensional equations of motion.

2. New Classical Action Principle

Classical mechanics assumes points which trace out linear paths. The equations of motion are based upon minimizing the distance of the path. String theory introduces one-dimensional objects which trace out areas, and the equations of motion are analogously based upon minimizing the total area. Membrane theory proposes two-dimensional objects which trace out (three-dimensional) volumes to be minimized. Our new action principle is that we should add all of these contributions together, and treat particles as polygeometric objects which trace out polydimensional paths with (26b) the quantity to be minimized.

Using only the vector and bivector contributions of (26b) the Lagrangian that is analogous to (18) would be,

$$\mathcal{L}(x^\mu, \mathring{x}^\mu, a^{\alpha\beta}, \mathring{a}^{\alpha\beta}) = m_0 c \sqrt{\mathring{x}^\mu \, \mathring{x}^\nu \, g_{\mu\nu} - \frac{1}{2\lambda^2} \, \mathring{a}^{\alpha\beta} \, \mathring{a}^{\mu\nu} \, g_{\alpha\mu} \, g_{\beta\nu}} \; , \tag{27}$$

where the open dot denotes differentiation with respect to the new affine parameter $d\kappa$ (whereas the small dot with respect to the proper time $d\tau$),

$$\mathring{Q} \equiv \frac{dQ}{d\kappa} = \dot{Q} \frac{d\tau}{d\kappa} \; . \tag{28}$$

The relationship of the new affine parameter to the proper time is is a new *spin dilation factor* analogous to the Lorentz dilation factor (17b). Dividing (26b) by $d\tau$ or $d\kappa$, and noting $d\tau^2 \equiv dx^\mu \, dx_\mu$, this spin dilation factor is,

$$\frac{d\tau}{d\kappa} \equiv \left(1 - \frac{\dot{a}^{\mu\nu} \, \dot{a}_{\mu\nu}}{2c^2\lambda^2} \right)^{-1/2} = \sqrt{1 + \frac{\mathring{a}^{\mu\nu} \, \mathring{a}_{\mu\nu}}{2c^2\lambda^2}} \; . \tag{29}$$

Note this implies the magnitude of the bivector velocity with respect to the proper time (proportional to spin angular velocity) is bounded by $\lambda \, c$, just as linear velocity cannot exceed c.

3. Canonical Momenta

In analogy to eq. (19a) we interpret the spin to be the canonical momenta conjugate to the bivector coordinate,

$$\texttt{Spin:} \quad S_{\mu\nu} \equiv \lambda^2 \frac{\delta \mathcal{L}}{\delta \mathring{a}^{\mu\nu}} = m_0 \, \mathring{a}_{\mu\nu} = m \, \dot{a}_{\mu\nu} \; , \tag{30a}$$

$$\texttt{Momentum:} \quad p_\mu \equiv \frac{\delta \mathcal{L}}{\delta \overset{\circ}{x}{}^\mu} = m_0 \overset{\circ}{x}_\mu = m \, \dot{x}_\mu \ , \tag{30b}$$

$$\texttt{Dynamic Mass:} \quad m \equiv m_0 \overset{\circ}{\tau} = m_0 \frac{d\tau}{d\kappa} \ . \tag{30c}$$

For our Lagrangian (27), these satisfy the Dixon equation (15). When these momenta are reparameterized in terms of the more familiar proper time, they have spin enhanced mass defined by (30c), which is equivalent to eq. (16a).

Our Lagrangian (27) is invariant under the polydimensional coordinate rotation (between vectors and bivectors), generated by four the arbitrary parameters $\delta\phi^\alpha$ of the automorphism transformation analogous to eq. (8),

$$\delta x^\alpha = \lambda^{-1} \, a^\alpha_{\ \ \mu} \, \delta\phi^\mu \ , \tag{31a}$$

$$\delta a^{\mu\nu} = x^\mu \, \delta\phi^\nu - x^\nu \, \delta\phi^\mu \ . \tag{31b}$$

Noether's theorem associates with this symmetry transformation a new set of constants of motion,

$$Q_\mu = \frac{\delta\mathcal{L}}{\delta \overset{\circ}{x}{}^\alpha} \frac{\delta x^\alpha}{\delta\phi^\mu} + \frac{1}{2} \frac{\delta\mathcal{L}}{\delta \overset{\circ}{a}{}^{\alpha\beta}} \frac{\delta a^{\alpha\beta}}{\delta\phi^\mu} = a_\mu^{\ \ \alpha} \, p_\alpha + S_{\mu\beta} \, x^\beta \ . \tag{32}$$

Taking the derivative of (32) with respect to the affine parameter yields the familiar Weysenhoff condition, that the spin is pure spacelike in the rest frame of the particle,

$$p_\mu \, S^{\mu\nu} = 0 \ . \tag{33}$$

This is quite significant, because usually (33) is imposed at the onset by fiat, while we have provided an actual derivation based on the new automorphism symmetry of the Lagrangian!

C. Polyrotational Quasi-Coordinates and Quantization

We propose the body frame coordinates may be rotated by a generalized geometamorphic transformation, a combination of (8) and (21b) which leaves the Lagrangian (27) invariant. This provides clues as to the nature of the derivative with respect to the bivector coordinate. With this we can define a new 'spin operator' and generalized quantum wave equations based on (15).

1. Polydimensionally Rotating Coordinate Frames

We propose a polydimensional generalization of Section III.A.3, invoking our principles of polydimensional isotropy and relative dimensionalism discussed in Section II above. The poly-rotation operator is,

$$\mathcal{R}(\kappa) \equiv \exp\bigl[-(1/4) \, \hat{\mathbf{e}}_{\mu\nu} \Theta^{\mu\nu}(\kappa) - (1/2) \, \hat{\mathbf{e}}_\alpha \Phi^\alpha(\kappa)\bigr] \ . \tag{34a}$$

We propose that the *polyvelocity* is defined in analogy to eq. (22),

$$\overset{\circ}{\mathcal{Q}} \equiv \mathcal{M}/m_0 \equiv -2\lambda \, \mathcal{R}^{-1} \, \overset{\circ}{\mathcal{R}} \equiv \overset{\circ}{x}{}^\mu \hat{\mathbf{e}}_\mu + (2\lambda)^{-1} \, \overset{\circ}{a}{}^{\mu\nu} \hat{\mathbf{e}}_{\mu\nu} \ , \tag{34b}$$

where $\{x^\mu, a^{\alpha\beta}\}$ must therefore be anholonomic quasi-coordinates (as opposed to the holonomic coordinates $\lambda\Phi^\mu$ and $\lambda^2\Theta^{\alpha\beta}$ respectively), such that their derivatives with respect to $d\kappa$ yield the vector and bivector velocities. In analogy to the rotational development in Section III.A.3 above, the linear velocity is NOT the (total) derivative of the usual cartesian coordinate $X^\mu(\kappa)$ for a spinning body. For example, when moving in the y direction, an angular acceleration of the spin along the z axis will introduce an additional "effective" velocity in the x direction due to the shift in apparent relativistic mass center. All of these interdependent effects are accounted for in the history-dependent quasi-coordinate x^μ.

It follows that the tangent basis vectors are given in analogy to eq. (23b),

$$2\lambda\frac{\partial\mathcal{R}}{\partial x^\mu} = -\mathbf{e}_\mu\mathcal{R} \ = -\mathcal{R}\hat{\mathbf{e}}_\mu \ , \tag{35a}$$

$$2\lambda^2\frac{\partial\mathcal{R}}{\partial a^{\alpha\beta}} = -\mathbf{e}_{\alpha\beta}\mathcal{R} = -\mathcal{R}\hat{\mathbf{e}}_{\alpha\beta} = 2\lambda^2\left[\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial x^\beta}\right]\mathcal{R} \ . \tag{35b}$$

This implies that the bivector derivative is equivalent to the commutator derivative, an idea developed further by Erler[7] and utilized in Section IV below. Two other commutators we shall find useful are,

$$\left[\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial a^{\alpha\beta}}\right]\mathcal{R} = -\left(\frac{g_{\mu\sigma}}{\lambda^2}\right)\delta^{\omega\sigma}_{\alpha\beta}\frac{\partial}{\partial x^\omega}\mathcal{R} \ , \tag{35c}$$

$$\left[\frac{\partial}{\partial a^{\alpha\beta}},\frac{\partial}{\partial a^{\mu\nu}}\right]\mathcal{R} = -\left(\frac{g^{\phi\kappa}}{\lambda^2}\right)g_{\sigma\mu}\,g_{\omega\nu}\,\delta^{\sigma\omega\theta}_{\alpha\beta\kappa}\,\frac{\partial}{\partial a^{\theta\phi}}\mathcal{R} \ . \tag{35d}$$

We note in passing that the 4 basis vectors and 6 bivectors of a 4D space can be reinterpreted as the 10 bivectors of an enveloping 5D space, with eq. (34a) as the rotation operator. In this context our calculus may be related to that proposed by Blake[2] over the multivector manifold $spin^+(4,1)$.

By parallel argument to eq. (25ab) we obtain the non-commutivity of the variation and derivative for the polyvelocity,

$$(\delta\mathbf{d} - \mathbf{d}\delta)\,\mathcal{Q} = \mathcal{R}^{-1}\left(\mathbf{d}\mathcal{R}\,\delta\mathcal{Q} - \delta\mathcal{R}\,\mathbf{d}\mathcal{Q}\right) \ = \ \left[\mathbf{d}\mathcal{Q},\delta\mathcal{Q}\right]/(2\lambda^2) \ . \tag{36}$$

Extracting the vector and bivector portions,

$$(\delta d - d\delta)\,x^\alpha = \left(\delta x^\mu\,da_\mu{}^\alpha - dx^\mu\,\delta a_\mu{}^\alpha\right)/\lambda^2 \ , \tag{37a}$$

$$(\delta d - d\delta)\,a^{\mu\nu} = \left(da^{\mu\sigma}\delta a_\sigma{}^\nu - \delta a^{\mu\sigma}da_\sigma{}^\nu\right)/\lambda^2 + \left(dx^\mu\delta x^\nu - \delta x^\mu dx^\nu\right) \ . \tag{37b}$$

Comparing to (20d) and (25b) we see that $\overset{\circ}{a}\simeq\omega\lambda^2$. However, we have additional terms which validates that variations of the linear and rotational paths are not independent, hence eq. (20c) is no longer complete.

2. Quantization

In classical Hamilton mechanics, functions of motion (n.b. the Hamiltonian on which quantum mechanics is based) are parameterized in terms of the coordinates and their canonical momenta. The obvious generalization of the Poisson Bracket for two functions of polydimensional coordinates would be,

$$\{F,G\} \equiv \left(\frac{\delta F}{\delta x^\alpha}\frac{\delta G}{\delta p_\alpha} - \frac{\delta G}{\delta x^\alpha}\frac{\delta F}{\delta p_\alpha}\right) + \frac{\lambda^2}{2}\left(\frac{\delta F}{\delta a^{\alpha\beta}}\frac{\delta G}{\delta S_{\alpha\beta}} - \frac{\delta G}{\delta a^{\alpha\beta}}\frac{\delta F}{\delta S_{\alpha\beta}}\right) \ . \tag{38}$$

There are some potential complications. From eq. (37ab) its not at all clear that δp^σ is completely independent of δx^μ, $\delta a^{\alpha\beta}$ and especially $\delta S^{\alpha\beta}$. For today we will sidestep the issues and assume for brevity that at least the canonical pairs obey the relations: $\{x^\alpha, p_\beta\} = \delta^\alpha_\beta$, and $\{a^{\mu\nu}, S_{\alpha\beta}\} = \lambda^2\delta^{\mu\nu}_{\alpha\beta}$.

The *Heisenberg quantization rule* is that the commutator of quantum operators maps to the Poisson bracket of the corresponding classical quantities,

$$[\hat{F}, \hat{G}] \mapsto i\hbar\{F,G\} \ . \tag{39}$$

It follows that $[\hat{x}^\nu, \hat{p}_\mu] = i\hbar\delta^\nu_\mu$ and $[\hat{a}^{\mu\nu}, \hat{S}_{\alpha\beta}] = i\hbar\lambda^2\delta^{\mu\nu}_{\alpha\beta}$. In the coordinate representation the momenta operators must be,

$$\hat{p}_\mu \equiv -i\hbar\frac{\partial}{\partial x^\mu} \ , \qquad \hat{S}_{\mu\nu} \equiv -i\hbar\lambda^2\frac{\partial}{\partial a^{\mu\nu}} \ . \tag{40ab}$$

This would imply that one could define a spin angular coordinate $\theta^{\mu\nu} = \lambda^{-2}\, a^{\mu\nu}$. We should note that other authors see potential difficulties with the definition of angular operators in quantum mechanics[19].

3. (Hand) Wave Equations

The polydimensional analogy of the Klein-Gordon wave equation based on the Dixon equation (15) would hence be,

$$\left[\hat{p}^\mu\,\hat{p}_\mu - \frac{1}{2\lambda^2}\hat{S}^{\mu\nu}\,\hat{S}_{\mu\nu}\right]\psi = -\hbar^2\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} - \frac{\lambda^2}{2}\frac{\partial}{\partial a^{\mu\nu}}\frac{\partial}{\partial a_{\mu\nu}}\right]\psi = (m_0 c)^2\psi \ , \tag{41a}$$

where the wavefunction $\psi(x^\mu, a^{\alpha\beta})$ depends upon the vector position <u>and</u> bivector spin coordinates. If the system is in an eigenstate of total spin, then eq. (41a) simply reduces to the standard Klein-Gordon equation with spin-enhanced mass given by eq. (16a).

One might expect that the generalization of the Dirac equation would simply involve factoring (15) with the polymomenta (12b) into the linear form: $\hat{\mathcal{M}}\Psi = m_0 c\Psi$. Its not quite that simple because we know the components of the standard spin operator $\hat{\mathbf{S}} = (1/2)\hat{S}^{\alpha\beta}\mathbf{e}_{\alpha\beta}$ do not commute. Consistent with (35d), we have the standard relations[12],

$$\hat{\mathbf{S}}\hat{\mathbf{S}} = -(1/2)\hat{S}^{\mu\nu}\hat{S}_{\mu\nu} + \mathbf{e}_{\mu\nu}\mathbf{e}_{\alpha\beta}[\hat{S}^{\mu\nu}, \hat{S}^{\alpha\beta}] = -(1/2)\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} - 2i\hbar\lambda^{-2}\mathbf{S} \ . \tag{42a}$$

Equation (35b) implies that the components of the momenta no longer commute: $[\hat{p}_\mu, \hat{p}_\nu] = -i\hbar \hat{S}_{\mu\nu}/\lambda^2$, such that the square of the momentum vector $\hat{\mathbf{p}} = \hat{p}^\mu \mathbf{e}_\mu$,

$$\hat{\mathbf{p}}\hat{\mathbf{p}} = \hat{p}^\mu \hat{p}_\mu + \frac{1}{2}\mathbf{e}_{\mu\nu}\,[\hat{p}^\mu, \hat{p}^\mu] \;=\; \hat{p}^\mu \hat{p}_\mu - i\hbar\lambda^{-2}\hat{\mathbf{S}} \;. \tag{42b}$$

Equation (35c) implies that the spin and momenta operators do not commute,

$$[\hat{p}^\mu, \hat{S}_{\alpha\beta}] = -i\hbar\delta^{\mu\sigma}_{\alpha\beta}\,\hat{p}_\sigma \;, \tag{42c}$$

$$\{\hat{\mathbf{p}}, \hat{\mathbf{S}}\} = 2\hat{\mathbf{p}} \wedge \hat{\mathbf{S}} - 3i\hbar\lambda^{-1}\hat{\mathbf{p}} \;. \tag{42d}$$

Putting eq. (42abd) together, the polymomenta operator (12b) obeys,

$$\widehat{\mathcal{M}}\widehat{\mathcal{M}} = \left(\hat{p}^\mu\,\hat{p}_\mu - \frac{1}{2\lambda^2}\hat{S}^{\mu\nu}\,\hat{S}_{\mu\nu}\right) - \frac{3i\hbar}{\lambda}\widehat{\mathcal{M}} + \frac{2}{\lambda}\hat{\mathbf{p}} \wedge \hat{\mathbf{S}} \;. \tag{43}$$

Substituting, we can rewrite eq. (41a) as,

$$\left[\widehat{\mathcal{M}}\left(\widehat{\mathcal{M}} + \frac{3i\hbar}{\lambda}\right) - \frac{2}{\lambda}\hat{\mathbf{p}} \wedge \hat{\mathbf{S}} - (m_0 c)^2\right]\psi = 0 \;. \tag{41b}$$

If we presume an idempotent structure on the wavefunction, the trivector term can be replaced by an eigenvalue,

$$\frac{2}{\lambda}\hat{\mathbf{p}} \wedge \hat{\mathbf{S}}\left[\left(1 \pm \frac{i\hat{\mathbf{p}} \wedge \hat{\mathbf{S}}}{mcS}\right)\psi\right] \;\Rightarrow\; \pm\left(\frac{2imS}{\lambda}\right)\psi \;, \tag{44}$$

where $S \equiv \parallel \mathbf{S} \parallel = (1/2)S^{\mu\nu}S_{\mu\nu}$. Thus eq. (41b) can now be factored into a polydimensional monogenic Dirac equation with complex mass roots $N_\pm$,

$$\left[\widehat{\mathcal{M}} + N_\pm\right]\Psi = \left[-i\hbar\left(\mathbf{e}^\mu\frac{\partial}{\partial x^\mu} + \frac{\lambda}{2}\mathbf{e}^{\mu\nu}\frac{\partial}{\partial a^{\mu\nu}}\right) + N_\pm\right]\Psi = 0 \;. \tag{45a}$$

$$\Psi \equiv \left[\widehat{\mathcal{M}} - N_\mp\right]\psi \;. \tag{45b}$$

Solving the quadratic equation, we can get one of the roots to be the bare mass if we impose a constraint on the magnitude of spin,

$$N_+ = m_0 c \;, \qquad N_- = m_0 c - 3i\hbar/\lambda \;, \tag{45cd}$$

$$S \equiv (1/2)S^{\mu\nu}\,S_{\mu\nu} \;= (3/2)\hbar(m_0/m) \;. \tag{45e}$$

Invoking eq. (16a), we can thus get a relationship between the fundamental constants m_0, λ and the magnitude of the spin S. In the limit of $m_0 c\lambda >> \hbar$ one recovers the standard 'half integer spin' magnitude equation: $S^2 = (3/2)\hbar^2$.

IV. General Poly-Covariance

In curved space, particles will now deviate from standard geodesics due to contributions from derivatives of the basis vectors with respect to the new bivector coordinate. Further, there are additional contributions to the non-commutivity of the variation and derivative due to torsion and curvature. This leads to a new derivation of the Papapetrou equations[14] describing the motion of spinning particles in curved space. Finally we propose a principle of **Metamorphic Covariance:** *that the laws of physics should be form invariant under local automorphism transformations which reshuffle the geometry.*

A. Covariant Derivatives in the Clifford Manifold

The total derivative of a basis vector with respect to the new affine parameter $d\kappa$ must by the chain rule contain a derivative with respect to the bivector coordinate,

$$\overset{\circ}{\mathbf{e}}_\mu \equiv \frac{d\mathbf{e}_\mu}{d\kappa} = \overset{\circ}{x}{}^\sigma \frac{\partial \mathbf{e}_\mu}{\partial x^\sigma} + \frac{1}{2} \overset{\circ}{a}{}^{\alpha\beta} \frac{\partial \mathbf{e}_\mu}{\partial a^{\alpha\beta}} \ . \tag{46a}$$

Our ansätze, consistent with (35b), is that the bivector derivative obeys[7, 15],

$$\frac{\partial \mathbf{e}_\mu}{\partial a^{\alpha\beta}} \equiv \left([\partial_\alpha, \partial_\beta] - \tau^\sigma_{\alpha\beta}\partial_\sigma\right)\mathbf{e}_\mu = \left(R_{\alpha\beta\mu}{}^\nu - \tau^\sigma_{\alpha\beta}\Gamma^\nu_{\sigma\mu}\right)\mathbf{e}_\nu \ , \tag{46b}$$

where $\tau^\sigma_{\alpha\beta}$ is the torsion, $\Gamma^\nu_{\sigma\mu}$ the Cartan connection and $R_{\alpha\beta\mu}{}^\nu$ the Cartan curvature.

We can factor out the basis vectors by defining the covariant derivative,

$$\frac{\partial}{\partial x^\mu}\left(p^\nu \mathbf{e}_\nu\right) = \mathbf{e}_\nu \nabla_\mu p^\nu \equiv \mathbf{e}_\nu \left(\partial_\mu p^\nu + p^\sigma \Gamma^\nu_{\mu\sigma}\right) \ , \tag{47a}$$

$$\frac{\partial}{\partial a^{\alpha\beta}}\left(p^\nu \mathbf{e}_\nu\right) = \mathbf{e}_\nu \left[\nabla_\alpha, \nabla_\beta\right] p^\nu \equiv \mathbf{e}_\nu \left(R_{\alpha\beta\mu}{}^\nu p^\mu - \tau^\sigma_{\alpha\beta}\nabla_\sigma p^\nu\right) \ . \tag{47b}$$

From these definitions it is clear than the covariant derivatives of the basis vectors $\mathbf{e}^\mu$ and $\mathbf{e}_\mu$ vanish as usual.

The parallel transport of the conserved canonical momenta generates new *poly-autoparallels* in the Clifford manifold,

$$0 = \frac{d}{d\kappa}\left(\mathbf{e}_\mu p^\mu\right) = \mathbf{e}_\mu \left(\overset{\circ}{x}{}^\sigma \nabla_\sigma + 2^{-1} \overset{\circ}{a}{}^{\alpha\beta} \left[\nabla_\alpha, \nabla_\beta\right]\right) p^\mu \ , \tag{48a}$$

$$0 = \frac{d}{d\kappa}\left(\mathbf{e}_{\mu\nu} S^{\mu\nu}\right) = \mathbf{e}_{\mu\nu} \left(\overset{\circ}{x}{}^\sigma \nabla_\sigma + 2^{-1} \overset{\circ}{a}{}^{\alpha\beta} \left[\nabla_\alpha, \nabla_\beta\right]\right) S^{\mu\nu} \ . \tag{48b}$$

Substituting (47ab) provides a new derviation of the Papapetrou equations of motion for spinning particles[14] in contravariant form. Ours however are more general as they include torsion and all the higher order terms. In covariant form,

$$0 = \overset{\circ}{p}_\sigma - \left(\overset{\circ}{x}{}^\alpha \Gamma^{\ \mu}_{\alpha\sigma} + 2^{-1} \overset{\circ}{a}{}^{\alpha\beta} R'_{\alpha\beta\sigma}{}^\mu\right) p_\mu \ , \tag{49a}$$

$$0 = \overset{\circ}{S}_{\rho\omega} - \delta^{\sigma\nu}_{\rho\omega} \left(\overset{\circ}{x}{}^\alpha \Gamma^{\ \mu}_{\alpha\sigma} + 2^{-1} \overset{\circ}{a}{}^{\alpha\beta} R'_{\alpha\beta\sigma}{}^\mu\right) S_{\mu\nu} \ , \tag{49b}$$

$$R'_{\alpha\beta\nu}{}^\mu \equiv R_{\alpha\beta\nu}{}^\mu - \tau^\sigma_{\alpha\beta}\Gamma^{\ \mu}_{\sigma\nu} \ . \tag{49c}$$

B. An-Holonomic Mechanics

It has been a long-standing unsolved problem to derive the Papapetrou equations from a simple Lagrangian. We succeed where so many others have failed because of our definition of the new affine parameter, the form of the Lagrangian (27) and by noting that the introduction of the bivector coordinate has made the system an-holonomic. Consider the variation of the Lagrangian,

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta x^\alpha}\,\delta x^\alpha + \frac{\delta\mathcal{L}}{\delta \overset{\circ}{x}{}^\alpha}\,\delta \overset{\circ}{x}{}^\alpha + \frac{1}{2}\frac{\delta\mathcal{L}}{\delta a^{\alpha\beta}}\,\delta a^{\alpha\beta} + \frac{1}{2}\frac{\delta\mathcal{L}}{\delta \overset{\circ}{a}{}_{\alpha\beta}}\,\delta \overset{\circ}{a}{}^{\alpha\beta}\ . \tag{50}$$

As in (20b), we must integrate the spin-velocity term by parts,

$$\frac{\delta\mathcal{L}}{\delta \overset{\circ}{a}{}_{\alpha\beta}}\delta \overset{\circ}{a}{}^{\alpha\beta} = S_{\alpha\beta}\,\delta \overset{\circ}{a}{}^{\alpha\beta} = \frac{d}{d\kappa}\left(S_{\alpha\beta}\,\delta a^{\alpha\beta}\right) - \overset{\circ}{S}{}_{\alpha\beta}\,\delta a^{\alpha\beta} + S_{\alpha\beta}\left(\delta \overset{\circ}{a}{}^{\alpha\beta} - \frac{d}{d\kappa}\delta a^{\alpha\beta}\right). \tag{51}$$

However, the generalized equation of motion eq. (20c) is incomplete because in general there will be an interdependence between the vector and bivector variations in curved space. Certainly we saw this feature appear before in eq. (37ab) for the polyrotating coordinate system. The difficulty is how to derive the new contributions due to torsion and curvature.

Note while (25b) states that the variation of the angular velocity is not the velocity of the angular variation, *for the components*, one can easily show from (24ab) and (25ab) that $\mathbf{d}(\delta\omega^{\mu\nu}\mathbf{e}_{\mu\nu}) = \delta(d\omega^{\mu\nu}\mathbf{e}_{\mu\nu})$. We therefore argue that in curved space the same idea holds,

$$\delta\left(\overset{\circ}{x}{}^\mu\,\mathbf{e}_\mu\right) = \frac{d}{d\kappa}\left(\delta x^\mu\,\mathbf{e}_\mu\right)\ , \tag{52a}$$

$$\delta\left(\overset{\circ}{a}{}^{\alpha\beta}\,\mathbf{e}_\alpha\wedge\mathbf{e}_\beta\right) = \frac{d}{d\kappa}\left(\delta a^{\alpha\beta}\,\mathbf{e}_\alpha\wedge\mathbf{e}_\beta\right)\ . \tag{52b}$$

Performing the variations and derivatives in the above equations and rearranging terms [and ignoring the contribution of eq. (37ab)],

$$\left(\delta\overset{\circ}{x}{}^\mu - \frac{d\delta x^\mu}{d\kappa}\right) = \delta x^\alpha\,\overset{\circ}{x}{}^\beta\tau^\sigma_{\alpha\beta} + \frac{1}{2}\left(\delta x^\alpha\,\overset{\circ}{a}{}^{\mu\nu} - \overset{\circ}{x}{}^\alpha\delta a^{\mu\nu}\right)R'_{\mu\nu\alpha}{}^\sigma\ , \tag{53a}$$

$$\left(\delta\overset{\circ}{a}{}^{\mu\nu} - \frac{d\delta a^{\mu\nu}}{d\kappa}\right) = \delta^{\lambda\sigma}_{\omega\nu}\left[\Gamma^\omega_{\alpha\mu}\left(\overset{\circ}{x}{}^\alpha\delta a^{\mu\nu} - \overset{\circ}{a}{}^{\mu\nu}\delta x^\alpha\right) + \frac{1}{4}R'_{\alpha\beta\mu}{}^\omega\left(\overset{\circ}{a}{}^{\mu\nu}\delta a^{\alpha\beta} - \overset{\circ}{a}{}^{\alpha\beta}\delta a^{\mu\nu}\right)\right]\ . \tag{53b}$$

The first term on the right of (53a) involving the torsion follows Kleinert[13], the rest are new. Substituting (53a) into (20b) and (52b) into (51), and finally into (50), separating out terms proportional to δx^μ and $\delta a^{\mu\nu}$ respectively, we obtain polydimensionally generalized Euler-Lagrange equations,

$$\frac{\delta\mathcal{L}}{\delta x^\mu} - \overset{\circ}{p}{}_\mu + p_\lambda\ \overset{\circ}{x}{}^\alpha\,\tau^\lambda_{\alpha\mu} + \left(2^{-1}p_\lambda\,R'_{\alpha\beta\mu}{}^\lambda + S_{\omega\beta}\,\Gamma^\omega_{\mu\alpha}\right)\ \overset{\circ}{a}{}^{\alpha\beta} = 0\ , \tag{54a}$$

$$\frac{\delta\mathcal{L}}{\delta a^{\mu\nu}} - \overset{\circ}{S}{}_{\mu\nu} + p_\lambda\ \overset{\circ}{x}{}^\sigma R'_{\mu\nu\sigma}{}^\lambda + 2^{-1}\left(S_{\omega\beta}R'_{\mu\nu\alpha}{}^\omega - S_{\omega\nu}R'_{\alpha\beta\mu}{}^\omega\right)\overset{\circ}{a}{}^{\alpha\beta} = 0\ . \tag{54b}$$

The first two terms of eq. (54a) are standard, the third term appears in Kleinert[13], the rest of (54a) and all of (54b) are new. Explicitly performing the derivative on the Lagrangian in (54a) we recover the Papapetrou equation (49a). To get the spin equation (49b) from (54b) we must introduce a generalization of eq. (46b) for bivector variations,

$$\frac{\delta \mathcal{L}}{\delta a^{\mu\nu}} \equiv \left[\frac{\delta}{\delta x^{\mu}}, \frac{\delta}{\delta x^{\nu}}\right] \mathcal{L} - \tau^{\sigma}_{\mu\nu} \frac{\delta \mathcal{L}}{\delta x^{\sigma}} = \frac{\delta \mathcal{L}}{\delta g_{\alpha\beta}} \left(R'_{\mu\nu\alpha\beta} + R'_{\mu\nu\beta\alpha}\right) \ . \tag{54c}$$

If there is no torsion, the R' reduces to the Riemann curvature, which is antisymmetric in the last two indices, hence eq. (54c) vanishes.

C. Metamorphic Covariance

Our Lagrangian (27) is invariant under *local* automorphism transformations, where in general the Φ^{μ} of (34) can be position dependent upon a path-dependent (history dependent) integral of a gauge field $B^{\nu}{}_{\mu}$,

$$\Phi^{\nu}(x^{\alpha}) = \int^{x^{\alpha}} B^{\nu}{}_{\mu}(y^{\sigma})\, dy^{\mu} \ . \tag{55}$$

This would imply that the connection of a basis vector would become *geometamorphic*[16], e.g. under parallel transport *a vector will metamorph into a plane.* We have previously proposed[15] such a"metamorphic Clifford connection" of the form,

$$\mathcal{D}\mathbf{e}_{\mu} \equiv dx^{\alpha}\left(\Gamma^{\nu}_{\alpha\mu} + \frac{1}{2}\Xi_{\alpha\mu}{}^{\nu\sigma}\mathbf{e}_{\nu\sigma}\right) + \frac{1}{2}da^{\alpha\beta}\left(R_{\alpha\beta\mu}{}^{\nu}\mathbf{e}_{\nu} + \frac{1}{2}\Omega_{\alpha\beta\mu}{}^{\nu\sigma}\mathbf{e}_{\nu\sigma}\right) \ , \tag{56a}$$

where $\Xi_{\alpha\mu}{}^{\nu\sigma} \simeq B^{\omega}{}_{\alpha}\delta^{\nu\sigma}_{\omega\mu}$, and the curvature $R_{\alpha\beta\mu}{}^{\nu}$ now has contributions from derivatives on both $\Gamma^{\nu}_{\alpha\mu}$ and $\Xi_{\alpha\mu}{}^{\nu\sigma}$. This means that equations (48ab) are no longer valid because each only contains a <u>single</u> dimensional piece. We are forced to implement *dimensional democracy*, and write our equations only with *polyvectors*. Further one finds that the Leibniz rule does not hold over the wedge (or dot) product, although it is valid for the Clifford (direct) product[15]. Hence the metamorphic connection on the bivector would be computed,

$$\mathcal{D}(\mathbf{e}_{\mu}\wedge\mathbf{e}_{\nu}) = \frac{1}{2}[(\mathcal{D}\mathbf{e}_{\mu}), \mathbf{e}_{\nu}] + \frac{1}{2}[\mathbf{e}_{\mu}, (\mathcal{D}\mathbf{e}_{\nu})] \neq (\mathcal{D}\mathbf{e}_{\mu})\wedge\mathbf{e}_{\nu} + \mathbf{e}_{\mu}\wedge(\mathcal{D}\mathbf{e}_{\nu}) \ . \tag{56b}$$

With these generalizations, reworking Section IV.A, one can get a polycovariant generalization[15] of the Papapetrou equation (49a),

$$\mathring{p}^{\mu} + p^{\nu}\left(\mathring{x}^{\beta}\Gamma^{\mu}_{\beta\nu} + \frac{1}{2}\,\mathring{a}^{\alpha\beta}R_{\alpha\beta\nu}{}^{\mu}\right) + \frac{1}{2}S^{\omega}{}_{\sigma}\left(\mathring{x}^{\alpha}\Xi_{\alpha\omega}{}^{\mu\sigma} + \frac{1}{2}\,\mathring{a}^{\alpha\beta}\Omega_{\alpha\beta\omega}{}^{\mu\sigma}\right) = 0. \tag{57}$$

To derive eq. (57) from a Lagrangian requires us to make the theory fully covariant under *general* polydimensional coordinate transformations. This will cause

the quadratic form (26b) to acquire cross terms such that the Lagrangian would generalize to,

$$\mathcal{L} = m_0 c \sqrt{\overset{\circ}{x}{}^{\alpha} g_{\alpha\beta}\, \overset{\circ}{x}{}^{\beta} + \frac{1}{2}\, \overset{\circ}{x}{}^{\alpha} h_{\alpha\mu\nu}\, \overset{\circ}{a}{}^{\mu\nu} + \frac{1}{4m_0^2\lambda^4}\, \overset{\circ}{a}{}^{\alpha\beta}\, \mathcal{I}_{\alpha\beta\mu\nu}\, \overset{\circ}{a}{}^{\mu\nu}}\ , \tag{58}$$

where $\mathcal{I}$ plays the role of the relativistic moment of inertia tensor. This and the interdimensional metric $h_{\alpha\mu\nu}$ will cause the linear momenta not to be parallel to the velocity and spin momenta not parallel to bivector (angular) velocity.

Equation (56a) is the classical analog to the spin covariant covariant derivative for the Dirac equation derived from generalized automorphism transformations of the Dirac algebra by Crawford[4],

$$\nabla_\mu = \partial_\mu + i\left(eA_\mu + \gamma^5 a_\mu\right) + \gamma_\nu \left(\frac{1}{2} B^{\nu}{}_{\mu} + \gamma^5\, i\, b^{\nu}{}_{\mu}\right) + \frac{1}{2}\gamma_{\alpha\beta}\, C^{\alpha\beta}_{\ \ \mu}\ , \tag{59a}$$

$$\left(-i\hbar\gamma^\mu \nabla_\mu - mc\right)\psi = 0\ . \tag{59b}$$

The gauge field $B^{\mu}{}_{\sigma}$ is the same as in eq. (55). In the Dirac equation (59b), the usual momentum operator eq. (40a) has been replaced by the gauge-covariant derivative: $p_\mu \to -i\hbar\nabla_\mu$. To get the interacting form of the polydimensional Dirac equation (45a) we have suggested[17] that one need only additionally replace the spin operator (40b) with the commutator covariant derivative: $S_{\mu\nu} \to -i\hbar\lambda^2\left[\nabla_\mu, \nabla_\nu\right]$,

$$\left(-i\hbar\gamma^\mu\nabla_\mu - i\hbar\frac{\lambda}{2}\gamma^{\alpha\beta}\left[\nabla_\alpha, \nabla_\beta\right] - m_0 c\right)\Psi = 0\ . \tag{60}$$

Certainly one could include higher order triple commutator derivatives. In flat space with all but the electromagnetic gauge field A_μ suppresed in (59a), the bivector (commutator) derivative will introduce an anomalous magnetic moment interaction which provides a possible interpretation of the constant λ. It remains to be shown that an application of Ehrenfest's theorem to eq. (60) can recover the equation of motion (57), in anology to the derivation of the Papapetrou equation (49a) from (59b) by Crawford[5].

V. Summary

In introducing *Dimensional Democracy* we have given the bivector a coordinate and shown its utility in the treatment of the classical spinning particle problem. This system is invariant under "polydimensional" transformations which reshuffle geometry such that 'what is a vector' is *Dimensionally Relative* to the observer's frame. A fundamentally new action principle has been introduced which is *Polydimensionally Isotropic*. Generalized *Metamorphic Covariant* equations of motion and quantum wave equations have been derived which include curvature, torsion and spin. Most important, the principles proposed have potential broad math and physics applications beyond the examples in this paper.

VI. Acknowledgments

Financial support for attending the conference was provided by Santa Clara University, Department of Physics. Particular thanks to the *Bazaar Café* of San Francisco for providing the creative environment in which the majority of this work was completed, and T. Erler (UC Santa Barbara) for numerous discussions during the development of most of the basic ideas. We appreciate J. Crawford's (Penn. State Univ.) help in showing that some of the results could be rewritten using standard covariant derivatives.

References

[1] W. Baylis, *Electrodynamics, A Modern Geometric Approach* (Birkhäuser, 1999).

[2] S. Blake, "Calculus on the multivector manifold $spin^+(p,q)$," J. Math. Phys. **39**, 6106-6117 (1998).

[3] J.S.R. Chisholm and R.S. Farwell, "Clifford Approach to Metric Manifolds," in *Proceedings on the Winter School on Geometry and Physics, Srni, 6-13 January, 1990*, Supplemento di Reconditi del Circulo Matematico di Palermo, Serie **2**, no. 26 (1991), pp. 123-133.

[4] J. P. Crawford, "Local automorphism invariance: Gauge boson mass without a Higgs particle," J. Math. Phys., **35** 2701-2718 (1994).

[5] J. P. Crawford, "Spinor Matter in a Gravitational Field: Covariant Equations a la Heisenberg," Found .Phys., **28**, 457-470 (1998).

[6] W.G. Dixon, "Dynamics of extended bodies in general relativiy, I. Momentum and angular momentum," Proc. Roy. Soc. Lond. **A314**, 499-527 (1970).

[7] T.G. Erler, "The Mathematics of the Spinning Particle Problem", *Clifford Algebras and Their Applications in Mathematical Physics, Proceedings of the fifth conference, Ixtapa, Mexico 1999*, R. Ablamowicz and B. Fauser eds., (in preparation), preprint gr-qc/9912024.

[8] P. Fiziev and H. KIeinert, "New Action Principle for Classical Particle trajectories in Spaces with Torsion," Europhys. Lett. **35**, 241 (1996); hep-th/9503074

[9] D. T. Greenwood, *Classical Dynamics,* (Dover, 1997), p. 113 and pp. 174-5.

[10] D. Hestenes, *New Foundations for Classical Mechanics*, (Kluwer, 1990), p. 307.

[11] D. Hestenes, *Spacetime Algebra* (Gordon and Breach Publ., 1965).

[12] I.B. Khriplovich, "Particle with internal angular momentum in a gravitational field," Sov. Phys. JETP **69**, 217-219 (1989).

[13] H. Kleinert, "Nonabelian Bosonization as a Nonholonomic Transformation from Flat to Curved Field Space," Annals. Phys. **253**, 121-176 (1997); hep-th/9606065.

[14] A. Papapetrou, "Spinning test-particles in general relativity, I", Proc. Roy. Soc. London **A209**, pp. 248-258 (1951); "Equations of motion in General relativity," Proc. Phys. Soc. **64**, 57 (1951).

[15] W. Pezzaglia, "Physical Applications of a Generalized Clifford Calculus," in *Dirac Operators in Analysis (Pitman Research Notes in Mathematics, Number 394)*, J. Ryan and D. Struppa eds., (Longman Science & Technology, 1998), pp. 191-202; gr-qc/9710027.

[16] W. Pezzaglia, "Polydimensional Relativity, A Classical Generalization of the Automorphism Invariance Principle," in *Clifford Algebras and Their Applications in Mathematical Physics, Proceedings of fourth conference, Aachen 1996*, Dietrich, Habetha and Jank eds., (Kluwer 1998), pp. 305-317; gr-qc/9608052.

[17] W. Pezzaglia, "Polydimensional Supersymmetric Principles", submitted to the *International Journal of Theoretical Physics*; gr-qc/9909071.

[18] F. Rohrlich, *Classical Charged Particles*, (Addison-Wesley, 1965), pp. 204-205.

[19] A. C. de la Torre and J. L. Iguain, "Angle states in quantum mechanics", Am. J. Phys. **66**, 1115-1122 (1998).